Revealing Features from Pareto Optimal Choice

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July 22, 2025

Abstract

A decision maker (DM) defines alternatives as a set of *features* (namely, measurable characteristics). An external observer sees only the DM's choices from various feasible sets of features but lacks information about which features the DM attends to, how they are evaluated, or what decision procedure is used. Under what conditions can the observer identify the subset of features that matter to the DM? Our approach relies on relatively weak behavioural assumptions—notably, that the DM does not choose options that are Pareto dominated with respect to the relevant features. We characterise which collections of choice–feasible set pairs are informative about the DM's underlying representation. Our results apply in environments with minimal structure—without requiring functional form assumptions or price data—and can accommodate settings where even only a single choice is observed.

JEL codes: D81, D83, D91

Keywords: revealed preferences, choice, characteristics, Pareto optimality.

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1 Introduction

1.1 Motivation

A growing literature documents systematic deviations from rational choice behaviour.[1] Modelling such deviations poses significant challenges, as identical choice patterns may be consistent with multiple behavioural explanations. Even when behaviour can be represented as utility maximisation, the precise trade-offs encoded in utility functions may prove unstable or unreliable.[2] By contrast, the specific *characteristics* of alternatives that influence decision-making arguably represent a more stable component of the agent's evaluative process. It is in this sense that what induces an agent to consume a good are some of its technical and economic attributes, and what drives his vote for a political party are its positions on certain key issues.

This paper focuses, rather than on the intensity with which such characteristics are valued, on *whether or not* they are considered at all. Adopting a model-free approach, we investigate the determinants of what features the decision maker (DM) attends to—what "makes the agent tick". The dimensionality of possible characteristics is typically large, and often it is psychologically implausible to assume that the DM processes all of them. Instead, the agent's internal representation of options may rely on a sparse subset of features—a perspective aligned with the "sparsity" framework of Gabaix (2014).

The DM is assumed to focus on a subset of relevant features—measurable characteristics of the available options—and to employ them within an unobserved decision procedure to make choices. What can we learn about the relevant features from the DM's choices? This identification problem is of fundamental importance for understanding the underlying motivations behind consumption, investment, political alignment, and other decision contexts. Crucially, our aim is to achieve such identification while imposing minimal structure on the aggregation or decision rule employed by the DM.

To illustrate the core idea of our approach, consider a simplified scenario involving only two features, representing the funding pledged by political parties for two issues: the defence budget and the international aid budget. The example aims at illustrative clarity rather than empirical realism; as noted, our general framework is ideally applied to settings with many such features. In this setting, a party's platform is fully described by a pair (x_1, x_2) of real numbers, denoting increases to defence and aid expenditure,

[1] See Caplin (2023) for a comprehensive discussion.

[2] For example, Cautious Utility of Cerreia-Vioglio et al. (2024).

respectively. Three parties A, B and C propose, respectively, the following platforms:

$$x^A = (5,1) \quad x^B = (4,4) \quad x^C \quad = (1,5)$$

Each voter prefers more of the feature they value positively and less of the feature they view negatively, with intensity and aggregation unspecified. The question is whether we can infer which features are relevant for the supporters of each party. Even without knowing whether the features are valued positively or negatively, a vote for party B reveals that both features must enter the voter's evaluative criteria. This is because if only defence had been relevant, then the voter would have favoured A if an increase of the defence budget was a positive feature, since x^A maximises the feature. And the voter would have favoured C if that increase was a negative feature, since x^C minimises the feature. A parallel argument excludes the case where only the aid budget is relevant.

By contrast, a vote for A or for C provides no information about which features are relevant to the voter. Both platforms x^A and x^C can be interpreted as maximisers on the feasible set $X = \{x^A, x^B, x^C\}$ of some non-trivial weighted sum of *any* set of features, for a suitable choice of weights. Indeterminacy prevails also when the choice of any platform is made from the restricted feasible set $\{x^A, x^B\}$ or $\{x^B, x^C\}$. However, if we *separately* observe that the voter chooses x^B in both $\{x^A, x^B\}$ and $\{x^B, x^C\}$, then the earlier inference applies once more. The first choice rules out the possibility that only defence is relevant, and the second rules out the possibility that only aid is relevant. Together, these observations imply that both features must have influenced the voter's decision.

In this example, the information conveyed by the two reduced choice problems coincides with the information conveyed by their union. We will later explore how, and in what sense, this property extends to more general settings.

This example highlights two important facts:

1. *Some* choices from *some* feasible sets provide clear evidence about which features the DM considers relevant, while other choices from the same set may yield no such information; and in some cases an entire choice set may fail to contain any informative alternative.

2. Multiple observations from different feasible sets can jointly yield significant information about the DM's relevant features, even when none of the individual choices, taken in isolation, is informative.

The goal of this paper is to develop these intuitions into a general theoretical framework. We will fully characterise the collections of pairs (X, x^*) of a feasible set and an observed choice that reveal, partially or fully, the relevant features, under relatively weak assumptions on the DM's decision process.

1.2 Our setup

In our analysis, a DM is simply viewed as an entity that takes a set of feasible levels of features (points in a Euclidean space) as an input to output a choice. The *type* of a DM is defined as the subset of features that are relevant to their decision-making process. We impose minimal structure on the DM's decision procedure. Specifically, we assume only that each relevant feature is either positively or negatively valued by the DM, and we require that choices satisfy the following behavioural principle:

Admissibility: In any feasible set, an alternative x is not chosen by the DM if there is a different alternative y that has weakly more of all positive relevant features, and weakly less of all negative relevant features.

In other words, we just impose that, on each choice occasion, the selected alternative satisfies a form of strong Pareto optimality with respect to the DM's type. Note that neither the directions of improvement nor the dimensions in which Pareto optimality operates are known to the observer. The Admissibility assumption is intentionally weak: it places no constraints on consistency across different feasible sets, and is thus fully compatible with violations of standard axioms such as the Weak or Strong Axiom of Revealed Preference. As a result, the framework can accommodate behaviour that cannot be rationalised by the maximisation of a stable utility function over the entire domain of alternatives.[3]

It is also important to recall that, even when all features are relevant and positively valued, and even within a single feasible set, strongly Pareto optimal choices are not generally characterised by the maximisation of a linear objective. As recently shown by Che et al. (2024), such choices may instead arise from sequential maximisation or sequential Nash bargaining procedures. These mechanisms—though originally developed in strategic or interpersonal contexts—can be reinterpreted in an intrapersonal setting as forms of "behavioural" decision-making processes.[4]

[3]Given the generality of our framework—which is compatible with a wide range of specific decision models—we remain agnostic at this stage about the underlying cognitive mechanisms, such as attention or preference formation, that may drive the neglect of certain features.

[4]In this vein, see, e.g., De Clippel and Eliaz (2012) for an intrapersonal bargaining model that explains behavioural "anomalies."

1.3 Results

Our analysis starts with identification conditions (Sections 3 and 4) for the benchmark scenario where the analyst observes choices from convex sets. Nevertheless, we can provide the intuition for the conditions by revisiting our initial finite example. We saw that a choice between any pair of alternatives, such as x^B from $X = \{x^A, x^B\}$, could have been made by any type.[5] A choice like x^B is completely uninformative because there is only one feasible kind of "trade-off" between features when moving away from x^B: it is only possible to have a larger defence expenditure and a smaller aid expenditure, but not vice-versa. This is not enough to discriminate between types. A specific form of *richness* in feasible trade-offs is needed.[6] Identifiability is equivalent to ensuring a sufficient variety of feasible trade-offs between features when moving away from the choice. For example, in $X = \{x^A, x^B, x^C\}$ changing the choice from x^B to x^A ($(4,4) \to (5,1)$) would entail diminishing expenditure on aid while increasing that on defence, while changing it from x^B to x^C ($(4,4) \to (1,5)$) would entail the opposite trade-off. The feasible set at x^B is "rich" and ensures identification. On the other hand, if the choice was x^A, changing it to x^B ($(5,1) \to (4,4)$) and changing it to x^C ($(5,1) \to (1,5)$) would entail the same kind of trade-off. The feasible set at x^A is not rich, hence there is no identification.

While the finite case provides useful intuition, our core analysis applies to general convex feasible sets. For instance, in the initial political example, one may imagine that voters are able to choose from a continuum of platform combinations, a case that could be approximated in practice in a finely graduated questionnaire. In this setting, we formulate the richness of trade-offs through geometric conditions.

Specifically, we provide four "*Orthant conditions*" that apply to various cases according to whether there is a single observation or multiple ones, and to whether partial or full identification is sought. The conditions impose requirements on the set of feasible directions at the observed point: this set can be interpreted in economic terms precisely as indicating all the possible tradeoffs from choice. Partial identification conditions are in terms of the set of feasible directions, or unions thereof, *not* being contained within any orthant. In contrast, the conditions for full identification strengthen this requirement: they impose that the set of feasible directions, or unions thereof, should *contain* certain orthants. All these conditions make precise the richness requirements of the set of feasi-

[5] Indeed, x^B is admissible by assuming that the first feature is negative and is the only relevant one, making type $\{1\}$ possible; or that the second feature is positive and is the only relevant one, making type $\{2\}$ possible; or that both features are relevant, making type $\{1,2\}$ possible.

[6] Note that since we are working in a setting in which the features' evaluations are unknown, a trade-off may involve two or more features all increasing or all decreasing. Thus, trade-off is intended here as a joint movement of two or more features, not necessarily in opposite directions.

ble tradeoffs that apply in the various cases. The gain from additional observations arises from the new feasible tradeoffs that they create across different choice problems. What is more, we also find that a necessary condition for full identification is that the *dimension* of the feasible set is sufficiently high relative to the number of potentially relevant features.

Next, we investigate the structure of the set of possible types, i.e., those types that the analyst cannot rule out for any given observation. We show that the set of possible types for a given observation is closed under union. This allows the observer to infer new possible types from known ones. Conversely, any collection of types that is closed under union is the set of possible types for some observation from some feasible set (Theorem 6).

Finally, we take a look (Section 6) at two specific cases of interest: when the feasible set is a polytope, and when it is a finite collection of points. The richness conditions can take in these cases a more operational flavour.

Beside its theoretical value in solving the identifiability problem, we hope that the present work can be of use in other specific ways. For example, consider the *design* of experiments, political polls, market surveys, and recommendation algorithms.[7] In all these cases the designer controls or influences the feasible set from which DMs choose. Our results can be of help to the designer in tailoring the set of feasible choices to increase its informativeness. Another setting where our work may prove useful is for screening irrelevant features prior to the application of a fully specified parametric model—a topic to which we return in the concluding section.

2 Framework and definitions

Let $F = \{1, \ldots, N\}$ be the set of possible *features*. A DM is observed to make choices on several occasions. The data are pairs of feasible sets and choice observations, $\mathcal{O} = \{(X_t, x^t)\}_{t \in T}$, where T is a finite index set, each $X_t \subseteq \mathbb{R}^N$ is a set of feasible alternatives and x^t, with $x^t \in X_t$ for all t, is the choice made by the DM on occasion t. An alternative $x = (x_1, ..., x_N) \in X_t$ is described by the amounts x_i, for all features $i \in F$, of that alternative.[8] Unless otherwise stated, the feasible sets X_t are nonempty closed bounded convex sets of $\mathbb{R}^N$.

We denote the boundary of a set $X \subseteq \mathbb{R}^N$ by ∂X, the origin by $\mathbf{0}$, the vectors of ones by $\mathbf{1}$, and the unit vector with i^{th} component equal to one by $\mathbf{1}_i$.

[7]On social media platforms algorithms learn what "features" consumers are interested in by presenting users with different "menus".

[8]Here, F is interpreted as the set of all conceivable features that could reasonably describe an alternative. Hence, the DM by assumption cannot use features that are not in F.

We now introduce our core behavioural assumption.

Definition 1. An *evaluation function* is a function $e : F \to \{-1, 0, 1\}$ such that $e(i) \neq 0$ for some $i \in F$.

Definition 2. A point $x \in X^t$ is *e-admissible* if

$$\left(\forall i : e(i)\, y_i \geq e(i)\, x_i\right) \text{ and } \left(\exists i : e(i)\, y_i > e(i)\, x_i\right) \text{ implies } y \notin X.$$

The function e is interpreted as indicating whether or not a feature is ignored; and if not, whether it is valued positively or negatively. Hence, e-admissibility means that x is Pareto optimal with respect to the evaluation function e. We have inbuilt in the definition the obvious assumption that not all features are ignored.

The DM's choice $x^t \in \partial X^t$ is observed for each t.[9] Our behavioural assumption is that there exists an (unobserved) evaluation function e such that the choice x^t is e-admissible for each t.

A *type* is a subset $I \subseteq F$. When the type is a singleton we call it *elementary*. Type I is *possible at* $\mathcal{O}$ if, for some hypothesis on the sign that the DM attaches to the features, we cannot find, on any choice occasion, any other feasible alternative that has weakly more of the positive features in I and weakly less of the negative features in I. Formally:

Definition 3. A type $I \subseteq F$ is *possible at* $x^t \in X^t$ if there exists an evaluation function e such that:

(i) $e(i) \neq 0$ if and only if $i \in I$;

(ii) x^t is e-admissible.

In this case, we also say that (e, I) is possible at x^t.

A type $I \subseteq F$ is *possible at* $\mathcal{O} = \left\{\left(X_t, x^t\right)\right\}_{t \in T}$ if there exists an evaluation function e such that (e, I) is possible at each $\left(X_t, x^t\right)$.

With this terminology, our question becomes whether, given a set of observations $\mathcal{O}$, we can exclude at least some type. If so, then the type is *partially identified*. If we can actually exclude all types except one, then the type is *fully identified*.[10] Formally:

Definition 4. The type is:

[9] A DM who cares about at least one feature always chooses x^t on the boundary of X_t. Hence, if an interior choice is observed, the model is falsified—perhaps, because set F misses a relevant feature.

[10] In some contexts it may be obvious whether some features (e.g., the price of a commodity) are positive or negative. As will be apparent, our analysis can be straightforwardly adapted to take into account these constraints (through restrictions on the evaluation function), which facilitate identification.

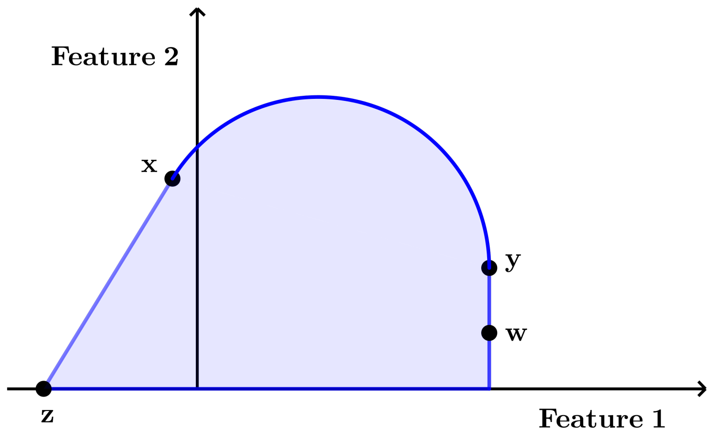

Figure 1: Possibility and identification: the type is fully identified at x and w, only partially identified at y, and not identified at z

- *partially identified* at $\mathcal{O}$ if some type is possible at $\mathcal{O}$ and there exists a type that is not possible at $\mathcal{O}$.
- *fully identified* at $\mathcal{O}$ if there exists exactly one type that is possible at $\mathcal{O}$.
- *not identified* at $\mathcal{O}$ if all types are possible at $\mathcal{O}$.

When $\mathcal{O}$ consists of a single observation (X, x) we will use the terminology "partially identified at x" instead of "partially identified at $\mathcal{O}$", and analogously for full and no identification.

We illustrate these notions in Figure 1. The feasible set is the convex set in blue. The only possible type at x considers both features (with feature 2 a good and feature 1 a bad). At w, type $\{1\}$ is possible with $e(1) = 1$ and $e(2) = 0$, and is the only possible type. This type remains possible at y, but now type $\{1, 2\}$ with $e(1) = e(2) = 1$ becomes possible too. Since feature 2 can be both increased and decreased, type $\{2\}$ can surely improve and thus it is not possible. Finally, at z, all types are possible considering features as negatives.[11]

In some, but not all, cases, possibility at some x^t may be linked to linear optimisation. Let $\langle ., . \rangle$ denote the inner product operation. We say that a type I is *linear* at x^t if there exists an $a \in \mathbb{R}^N$ such that $a_i \neq 0$ if and only if $i \in I$, and $\langle a, x^t \rangle \geq \langle a, x \rangle$ for all $x \in X_t$. In other words, a linear type at x^t is such that X_t is supported at x^t by a hyperplane whose non-zero coefficients correspond exactly with the features in I. In Figure 1, type $\{1\}$ is

[11] As the example suggests, in many cases, identifying the type automatically reveals the evaluation function. However, in this paper, we will focus on whether features matter, rather than whether they have a positive or negative impact.

linear at y maximising a linear function with direction $(1, 0)$, but type $\{1, 2\}$ is not, since no hyperplane with coefficients that are all non-zero supports the feasible set at y. Hence, the set of linear types can be a strict subset of the set of possible types.

Appendix B contains standard definitions and facts from convex analysis. Here we only report some definitions:

The *cone of feasible directions* of a set S at $x \in S$ is defined as

$$F_S(x) = \{\alpha(y - x) \,|\, y \in S, \, \alpha \geq 0\}.$$

This is the set of all directions in which it is possible to move away from x while remaining locally within the set S. From an economic perspective, the feasible directions tell us the feasible trade-offs between features at a given observation.

A convex cone C is *generated* by vectors $v^1, \ldots, v^\eta \in \mathbb{R}^N$, when $x \in C$ if and only if there exist numbers $\alpha_i \geq 0$ such that $x = \sum_{i=1}^{\eta} \alpha_i v^i$. Note that C is closed. An *orthant* of $\mathbb{R}^N$ is a convex cone generated by $v^1, ..., v^N \in \mathbb{R}^N$, where $v^i_j \in \{-1, 1\}$ if $j = i$ and $v^i_j = 0$ otherwise. A *k-dimensional orthant of* $\mathbb{R}^N$, $k \in \{1, \ldots, N\}$, is a convex cone generated by $v^{i_1}, \ldots, v^{i_k} \in \mathbb{R}^N$ such that $i_1, \ldots, i_k \in \{1, \ldots, N\}$ are different indices. For example, an N-dimensional orthant of $\mathbb{R}^N$ is a usual orthant, whereas a 1-dimensional orthant of $\mathbb{R}^N$ is a semi-axis.

As a preliminary, we provide the following "existence" result:

Proposition 1. *Let X be a convex compact subset of $\mathbb{R}^N$. Then:*

(i) For any type I there exists $x \in \partial X$ such that I is possible at x.

(ii) For any $x \in \partial X$ there exists a type I that is possible at x.

Part (i) shows that no type can be ruled out based solely on the feasible set prior to observing the DM's choice.[12] The second part of the proposition shows that any observation on the boundary of a feasible set X is compatible with the model. Convexity is essential: it is easy to construct a non-convex set with observations at which *no* type is possible.

All proofs are relegated to Appendix A.

3 Partial identification

Partial identification plays a central role in many applications. For instance, in a financial context where the features of an asset correspond to its past returns, an analyst may be

[12] Its proof is based on linear optimisation "discovering" the types. As noted previously, however, such a method will not necessarily discover *all* the types that are possible at each given observation. In Figure 1, y does not maximise any linear objective in the direction $(1, 1)$, but there is another feasible point where this direction is maximised.

(a) Partial identification ($\{2\}$ is ruled out), $F_X(x^*)$ intersects two orthants

(b) No identification, $F_X(x^*)$ is contained in a single orthant

Figure 2: The orthant condition for partial identification

interested in determining whether the DM ignores returns beyond a certain historical horizon—even if finer information about the evaluation process is not recoverable. In this section, we provide a characterisation of partial identifiability.

Here, and in the next section, it is convenient to distinguish the situation in which there is only a single choice occasion from the general case.

3.1 Single observation

The following result shows that, with only one observation, partial identifiability is equivalent to the feasible directions at the observed choice not being contained in an orthant. Recall that $F_X(x^*)$ denotes the cone of feasible directions at point x^*.

Theorem 1. ***(Orthant condition I: single observation, partial identification)*** *Suppose that there is only one occasion of choice, i.e.* $\mathcal{O} = \{(X, x^*)\}$ *for some* $X \subseteq \mathbb{R}^N$ *and* $x^* \in \partial X$*. Then the type is partially identified at* x^* *if and only if there is no orthant* K *of* $\mathbb{R}^N$ *such that* $F(x^*) \subseteq K$*.*

Figure 2 illustrates Orthant condition I. The feasible sets X are displayed in blue, the cones of feasible directions in green, and the normal cones in purple. The observed choice x^* is always made to coincide with the origin. Clearly, the analyst can rule out type $\{2\}$ in the left panel and cannot rule out any type in the right panel. The normal cone $N_X(x^*)$ contains all linear functions maximised at x^*. For example, $(1,0)$, $(0,1)$, and $(1,1)$ belong to $N_X(x^*)$ in the right panel, which means that all possible types are linear in this case.[13]

Orthant condition I illustrates that the richness of trade-offs required for identification does not simply mean that there are "sufficiently many" kinds of tradeoffs available, but

[13]More broadly, we show in Section 6 that all possible types are linear when X is a convex polytope.

rather that they must offer a "sufficient diversity" in terms of the signs of the rate of exchange that they imply.

To make this point more formally, it is true that when the set of feasible directions $F_X(x^*)$ is contained in some orthant, X must have a "sharp" shape around x^*—there are relatively few directions in which movement is possible. Intuitively, such limited flexibility reduces the informativeness of the observed choice. Conversely, if X is sufficiently "fat" around x^*—meaning there are many feasible directions—more types are excluded, enhancing identification.[14]

However, fatness is not a necessary condition for partial identification. Orthant condition I cannot be reduced to a simple lower bound on the local sparsity of alternatives. Even when there are "few" feasible tradeoffs—i.e., the feasible set has a sharp geometry—identification may still be achieved provided it is possible to execute a sufficient *diversity* of tradeoffs. For example, in Figure 2, the feasible set in the left panel is obtained from the one in the right panel by a rotation, yet it supports partial identification despite its sharpness around x^*. Here, starting from x^* it is possible both to reduce the horizontal feature while increasing the vertical feature, and to decrease both features. In the right-hand panel, only the latter movement is possible, and this makes the crucial difference for identification. Similarly, in Figure 3, a choice at a point where the feasible set is sharp rules out type $\{2\}$, thereby yielding partial identification.

The proof also shows:

Corollary 1. ***(Elementary types suffice)*** *The type is partially identified at x^* if and only if there exists an elementary type $\{i\} \subseteq F$ that is not possible at x^*.*

This result is interesting in two respects. First, since an elementary type is obviously linear, we see that partial identification is equivalent to the exclusion of some *linear* type. In other words, the statement and validity of Theorem 1 would be unchanged if the DM were assumed a priori to follow a linear maximisation rule over features. This will *not* hold for full identification.

[14]The idea of fatness can be made formally precise via the relationship between the tangent cone $T_X(x^*)$ and its dual cone $-T_X^0(x^*)$. Say that X is *fat at* $x^* \in X$ if $T_X(x^*) \nsubseteq -T_X^0(x^*)$. If there exists $x \in T_X(x^*)$ for which $x \notin -T_X^0(x^*)$, it means that there exists a $y \in T_X(x^*)$ which forms an obtuse angle with x. In other words, there is at least a direction in which the set $T_X(x^*)$ is fat (while it can be sharp in other directions). Note that the two-dimensional cases may be slightly misleading in this respect. In two dimensions, either $T_X(x^*) \subseteq -T_X^0(x^*)$ or $-T_X^0(x^*) \subseteq T_X(x^*)$, that is, either the set of feasible directions is unambiguously non-obtuse or it is unambiguously non-acute. But in higher dimensions, a cone may be thin in some directions and fat in others, so that neither of the dual containment relations holds. It is not difficult to see that the type is partially identified at any x^* at which X is fat. Suppose that $T_X(x^*)$ is contained in an orthant C_V. Then $\langle x, v^i \rangle \geq 0$ for each $x \in T_X(x^*)$ and each basis vector v^i. It follows that $v^i \in -T_X^0(x^*)$ for all v^i, and therefore $T_X(x^*) \subseteq C_V \subseteq -T_X^0(x^*)$. Hence, if X is fat at x^*, $T_X(x^*)$ cannot be contained in an orthant, and we can apply Theorem 1 to prove the assertion.

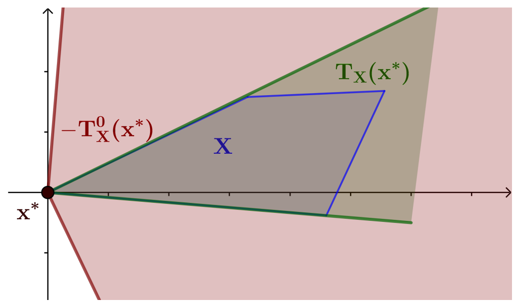

Figure 3: Sharpness with partial identification

Second, the "only if" direction of the corollary provides a powerful simplification for checking non-identifiability. Rather than considering the full class of possible types—which grows exponentially with the number of features, $2^N - 1$—one can restrict attention to just the N elementary types. If none of these can be excluded based on the observed choice, then no identification is possible. This reduction significantly enhances the practical tractability of partial identification tests, especially in high-dimensional settings. In Section 6 we develop this insight further by presenting an explicit matrix-based criterion for conducting such tests in the case where the feasible set X is a convex polytope. This yields a concrete and computationally efficient method for assessing partial identification in finite-dimensional decision problems.

3.2 General case

We are now ready to deal with partial identification in the general case.

To begin, note that the model is falsifiable by multiple observations, even when each observed choice lies on the boundary of its respective feasible set. For example, suppose that for two observations the feasible set is the same and coincides with the one depicted in Figure 1. If the observations are x and y, then no single type could have produced them. In fact, type $\{1,2\}$ is possible at each observation taken on its own, but in one case the evaluation function must treat feature 1 as a negative while in the other case it must treat it as a positive. Thus, no single sign assignment to the relevant features suffices to explain both choices simultaneously.

By contrast, consider a pair of observations drawn from different feasible sets. Suppose one observation is y, again from the feasible set in Figure 1, and the other is x^* from Figure 4b. In this case, type $\{1,2\}$ is consistent with both choices: in the first instance, types $\{1,2\}$ and $\{1\}$ are possible at y, while in the second type $\{1,2\}$ and $\{2\}$ are possible at x. Hence type $\{1,2\}$ is fully identified.

The key insight here is that additional observations contribute to partial identification by expanding the collection of feasible directions at the observed choices. The following result formalises this idea, extending the single-observation theorem by showing that a non-trivial collection of observations yields partial identification of the DM's type when the *union* of the feasible directions satisfies the non-inclusion condition.

Theorem 2. ***(Orthant condition II: multiple observations, partial identification)*** *Suppose that some type is possible at $\mathcal{O}$. Then the type is partially identified if and only if there is no orthant K of $\mathbb{R}^N$ such that $\bigcup_t F\left(x^t\right) \subseteq K$.*

4 Full identification

In this section, we demonstrate that a strengthening of the orthant-based conditions of the preceding analysis is sufficient to characterise full identification. In many cases, these conditions can be succinctly expressed as requiring that the interior of the cone of feasible directions augmented by the origin—or a union of such cones—*contains* an orthant. This geometric condition captures the idea that there is sufficiently rich variation in the feasible directions of movement at the observed choice point(s).

4.1 Single observation

Starting once again from the single-observation case, Figure 4 begins to guide intuition. As before, the feasible sets X are displayed in blue, the cones of feasible directions in green, the normal cones in purple, and the observed choice x^* coincides with the origin. Clearly, there are enough feasible tradeoffs around x^* in the left panel, which allows the analyst to rule out all types but one. In the right panel, however, the negative horizontal semi-axis is not in the interior of the cone of feasible directions. Here, the DM could have chosen x^* being interested either in both features 1 and 2, or only in feature 2.

The precise formulation of the orthant condition is, however, more subtle. Specifically, when the choice set X is full-dimensional, the feasible directions at x^* must form a cone that lies within a lower dimensional subspace—i.e., it involves only a subset of the available features—such that the interior of the cone *contains* an orthant of the subspace. In economic terms, this means that, in the subspace, tradeoffs of all kinds (increasing or decreasing a feature in exchange for increasing or decreasing another feature) are possible between the involved features. Moreover, the directions orthogonal to that subspace must also be feasible, making additional movements from the choice point available to the DM. Formally, we use the following terminology:

(a) Full identification (only $\{1,2\}$ is possible), *int* $F_X(x^*) \cup \{\mathbf{0}\}$ contains an orthant

(b) Partial identification (both $\{2\}$ and $\{1,2\}$ are possible), no orthant is contained in *int* $F_X(x^*) \cup \{\mathbf{0}\}$

Figure 4: The orthant condition for full identification, $\dim X = N$

Definition 5. *We say that there are* rich tradeoffs at x^* *if the following condition is satisfied*

$$F_X(x^*) \supseteq \left\{ x \in \mathbb{R}^N \mid \left(x_{i_1}, \ldots, x_{i_k}\right) \in K \right\}, \tag{1}$$

where $\{i_1, \ldots, i_k\} \subseteq F$ *are distinct indices,* $1 \leq k \leq N$*, and* K *is a closed convex cone in* $\mathbb{R}^k$ *such that int* $K \cup \{\mathbf{0}\}$ *contains an orthant of* $\mathbb{R}^k$*.*

This condition states an orthant requirement roughly "dual" to that of the condition for partial identification—in that it replaces "is not contained" with "contains"—with the difference that the contained orthant needs not involve all features and the containment is within the interior of the cone.

In Figure 4a, we can take $K = F_X(x^*)$ and the contained orthant is full-dimensional. To see why the condition must be stated in this way, allowing the contained orthant to not be full-dimensional, add to Figure 4a a third dimension, corresponding to feature 3. Figure 4a now depicts the cross-section at x^* of the feasible set X, which is a right triangular prism. From any feasible point in the diagram both movements up and down are possible. In particular, the choice point x^* lies in the interior of the edge connecting the two bases of the prism. Clearly, the cone of feasible directions contains all of the vectors depicted in green (cone K in the lower dimensional subspace, $k = 2$), as well as both directions of the feature 3 axis (the directions orthogonal to the subspace). Type $\{1,2\}$ is still fully identified in this 3-dimensional setting.

Now consider the case of a feasible set X whose dimension is one less than the number of features. In this case, for full identification the choice point x^* must be in the relative interior of X. In fact, if this condition holds, any direction opposite to a feasible one is also

(a) Full identification (only $\{2\}$ is possible), $x^* \in ri\, X$

(b) No identification (all types are possible), $x^* \notin ri\, X$

Figure 5: The interior condition for full identification, $\dim X = N - 1$

feasible. Hence, the choice of x^* fully reveals the DM's type because the only possible type is the linear type orthogonal to X. Conversely, if x^* were on the relative boundary of X, more types would become possible. Figure 5 compares two cases where X is a straight line segment in $\mathbb{R}^2$.

We can now characterise full identification using our two conditions:

Theorem 3. ***(Orthant condition III: single observation, full identification)*** *Suppose that there is only one occasion of choice, i.e.* $\mathcal{O} = \{(X, x^*)\}$ *for some* $X \subseteq \mathbb{R}^N$ *and* $x^* \in \partial X$. *Then the type is fully identified at* $x^* \in \partial X$ *if and only if either*

(i) $\dim X = N$ *and there are rich tradeoffs at* x^*

or

(ii) $\dim X = N - 1$ *and* $x^* \in ri\, X$.

To prove that Condition (i) is sufficient, we show in Appendix A that any type other than $I = \{i_1, \ldots, i_k\}$ can be ruled out. As a first step, (1) implies that both directions $\mathbf{1}_{i'}$ and $-\mathbf{1}_{i'}$ are feasible for any $i' \in F \setminus I$, so that the DM could have chosen something better than x^* if the type included such i'. Hence, the only possible types I' are such that $I' \subseteq I$. But the case $I' \subset I$ is also ruled out by the "contains an orthant" part of Condition (i). For the sufficiency of Condition (ii), we show similarly that any type other than the linear type orthogonal to X can be ruled out.[15]

[15] The necessity proof is more nuanced. If type I is fully identified, this means, in particular, that any linear type $I' \neq I$ is not possible. This imposes restrictions on the normal cone at x^*. Specifically, Lemma 3 in Appendix A shows that, in this case, the normal cone at x^* is either contained in the interior (plus the origin) of a possibly lower-dimensional orthant, or it is a 1-dimensional subspace. This in turn implies restrictions on the tangent cone at x^*. Lemma 4 demonstrates that either the analog of Condition (i) for the tangent cone at x^* holds, or Condition (ii) must be satisfied. What is left to prove is that we can replace the tangent cone in Condition (i) with the cone of feasible directions. Because the tangent cone is the closure

Theorem 3 says that full identification with one observation is only possible if the dimension of X is either N or $N-1$. Intuitively, if dim X is smaller, the DM's choice reveals too little about the features that drive it. Imagine, for example, a straight line segment in a 3-dimensional space. No matter how it is oriented and where x^* is located, the type cannot be identified. We record separately this immediate but notable implication, providing an absolute constraint on the feasible set for any hope of full identifiability:

Corollary 2. ***(Minimum dimensionality requirement)*** *Suppose that there is only one occasion of choice, i.e.* $\mathcal{O} = \{(X, x^*)\}$ *for some* $X \subseteq \mathbb{R}^N$ *and* $x^* \in \partial X$. *If* $\dim X < N-1$, *the type is fully identified for no* $x^* \in \partial X$.

An alternative characterisation in terms of normal cones, using similar arguments, is useful to make more transparent the dual task performed by Orthant condition III.

Theorem 4. ***(Normal cone condition)*** *Suppose that there is only one occasion of choice, i.e.* $\mathcal{O} = \{(X, x^*)\}$ *for some* $X \subseteq \mathbb{R}^N$ *and* $x^* \in \partial X$. *Then the type is fully identified at* $x^* \in \partial X$ *if and only if either*

(i) $N_X(x^*) \subseteq ri\, S \cup \{\mathbf{0}\}$ *for some k-dimensional orthant S of* $\mathbb{R}^N$, $1 \leq k \leq N$, *and* $x^* \in ri\, Y$, *where Y is a face of X parallel to* $N-k$ *coordinate axes with* $\dim Y \geq N-k$

or

(ii) $N_X(x^*)$ *is a* 1*-dimensional subspace of* $\mathbb{R}^N$.

In this characterisation, the part asserting that the normal cone is contained in an orthant guarantees the uniqueness of the possible linear type. On the other hand, the interiority condition on the face excludes the possibility of any non-linear type. Refer to Figure 1, where point y (partial identification) does not meet the interiority condition, whereas point w (full identification) does.

4.2 General case

Just like for partial identification, our analysis for a single observation provides implications on the way in which additional observations sharpen identifiability.

Let $\mathcal{F} = \bigcup_t F\left(x^t\right)$. In the same spirit as Definition 5, we introduce:

of $F_X(x^*)$, the desired condition for $F_X(x^*)$ does not follow automatically. First, we show that cone K can be obtained by slightly "stretching" the orthant contained in the tangent cone. The resulting K lies within the projection of $F_X(x^*)$ on $\mathbb{R}^k$ and $int\, K \cup \{\mathbf{0}\}$ contains an orthant of $\mathbb{R}^k$. Finally, we need to show that all directions orthogonal to the subspace of K are feasible. If that was not the case, convexity would imply that the feasible directions within the orthogonal subspace of dimension $N-k$ lie entirely within a closed half-space. This, in turn, would imply that a type $I' \supset I$ is possible too, a contradiction. Hence, all directions orthogonal to the subspace of K are feasible, and $F_X(x^*) \supseteq \left\{x \in \mathbb{R}^N \mid \left(x_{i_1}, \ldots, x_{i_k}\right) \in K\right\}$.

Definition 6. *We say that there are* rich tradeoffs at $\mathcal{O} = \left\{ \left(X_t, x^t \right) \right\}_{t \in T}$ *if the following condition is satisfied*

$$\mathcal{F} \supseteq \left\{ x \in \mathbb{R}^N \mid \left(x_{i_1}, \ldots, x_{i_k} \right) \in K \right\}, \tag{2}$$

where $\{i_1, \ldots, i_k\} \subseteq F$ *are distinct indices,* $1 \leq k \leq N$*, and* K *is a closed convex cone in* $\mathbb{R}^k$ *such that int* $K \cup \{\mathbf{0}\}$ *contains an orthant of* $\mathbb{R}^k$.

The interpretation is the same as before, except that now it applies to the union $\mathcal{F}$ of the sets of feasible directions across occasions. The following result generalises Theorem 3.

Theorem 5. ***(Orthant condition IV: multiple observations, full identification)*** *Suppose that some type is possible at* $\mathcal{O} = \left\{ \left(X_t, x^t \right) \right\}_{t \in T}$*. Then the type is fully identified at* $x^* \in \partial X$ *if either*

(i) $\dim \mathcal{F} = N$ *and there are rich tradeoffs at* $\mathcal{O}$

or

(ii) $\dim \mathcal{F} = N - 1$ *and* $\mathbf{0} \in ri\, \mathcal{F}$.

The conditions are necessary with $\mathcal{F}$ *replaced by* $conv(\mathcal{F})$.

An interesting difference with the single observation case—where the uniquely identified type must be linear—is that with multiple observations the uniquely identified type can be non-linear at *each* observation in $\mathcal{O}$. The example in Figure 6 illustrates. This fact underscores again how our behavioural assumption of admissibility goes beyond linear optimisation.

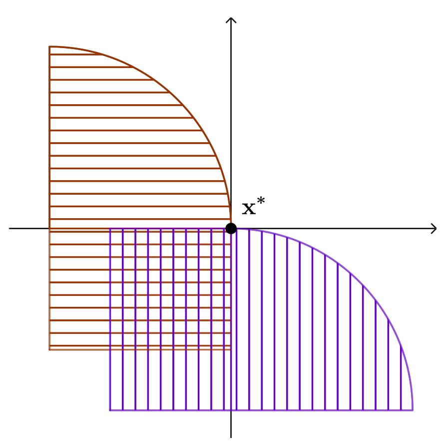

Figure 6: Type $\{1, 2\}$, which is non-linear (and thus not fully identifiable) at x^* in each observation taken on its own, is uniquely identified by the two observations taken together

5 On the structure of possibility

A natural question concerns the structure of the set of possible types. We show that this set possesses a closure property under union: if I and J are possible at some observation, then $I \cup J$ is also possible with an evaluation function that agrees with that of type I on features in I, and with that of type J on features in $J \setminus I$. The converse is also true: any collection of types that is closed under union can be rationalised by some observation. Moreover, such rationalisation can be achieved with a feasible set X that is a convex polytope. Formally:

Theorem 6. *(i) For all $X \subseteq \mathbb{R}^N$ and $x^* \in X$, the set of possible types at x^* is closed under union. (ii) For any nonempty collection $\mathcal{I} \subseteq 2^F \setminus \{\varnothing\}$ of types that is closed under union, there exists a convex polytope $X \subseteq \mathbb{R}^N$ and $x^* \in X$ such that the set of possible types at x^* is $\mathcal{I}$.*

In summary, a collection of types is the set of possible types at some observation if and only if it is closed under union. The first part of the result allows the analyst to focus on collections that satisfy this closure property and provides a constructive way to infer new admissible types from those already known. The second part establishes a negative result: beyond closure under union, no further structural constraints on the set of possible types can be imposed uniformly across all observations.

6 The linear case

6.1 Polytopes

In this section, we focus on the case in which the feasible set is a convex polytope. This is a particularly useful and economically relevant setting: many problems in economics can be cast as linear programming problems, and convex polytopes naturally arise from the convexification of finite sets of alternatives. Moreover, in this environment, every possible type is linear (Proposition 2), which brings a different perspective on the issue of identifiability.

We shall focus for simplicity on the single-observation case. Let X be a non-empty convex polytope (bounded polyhedron) in $\mathbb{R}^N$ defined by:

$$X = \{x \in \mathbb{R}^N \mid Bx \leq c\},$$

where B is an $m \times N$-matrix and c is a vector in $\mathbb{R}^m$. Let $B^{(j)}$ be the j-row of matrix B. Constraint j is *active* at $x^* \in X$ if $B^{(j)}x^* = c_j$. Denote by $\bar{B}(x^*)$ the matrix of constraints that are active at x^*.

The first result formalises the previous assertion about linear types:

Proposition 2. *If X is a convex polytope, type I is possible at $x^* \in \partial X$ if and only if I is linear.*

Proposition 2 allows us to focus on linear types in this section without loss of generality. Clearly, it is not difficult to prove that any linear type is a possible type. The proof of the converse (any possible type I is linear) is based on the application of the result in Arrow et al. (1953) to the projection of X on $\mathbb{R}^I$.

From now on we will denote a^I, $I \subseteq F$, any vector $a \in \mathbb{R}^N$ for which $a_i \neq 0$ if and only if $i \in I$. From Proposition 2 and the duality theory of linear programming, we can now derive the following characterisation of possible types in terms of the matrix of active constrains.

Proposition 3. *Type I is possible at $x^* \in \partial X$ if and only if there exist $a^I \in \mathbb{R}^N$ and $y \geq \mathbf{0}$ such that $a^I = \bar{B}^T(x^*)\, y$.*

In other words, type I being possible is equivalent to the existence of a coefficient vector a^I in the convex cone generated by the constraints that are active at x^*.

The previous result applies to any x^* on the boundary of the convex polytope X. In the remainder of this section, we will focus on the important special case when x^* is a vertex of X at the intersection of N hyperplanes with linearly independent normal vectors, i.e., a non-degenerate basic feasible solution of a linear program. In that case, $\bar{B}(x^*)$ is an invertible $N \times N$ matrix, which allows for a simpler criterion to tell whether or not type I is possible:

Corollary 3. *Suppose that $x^* \in \partial X$ and $\bar{B}(x^*)$ is an invertible $N \times N$ matrix. Then type I is possible at x^* if and only if there exists $a^I \in \mathbb{R}^N$ such that*

$$D(x^*)\, a^I \geq \mathbf{0} \tag{3}$$

for $D(x^*) = \left(\bar{B}^T(x^*)\right)^{-1}$.

The elements of $D(x^*)\, a^I$ characterise the change in $\langle a^I, . \rangle$ when moving from adjacent vertices to x^*. It follows from (3) that an elementary type is possible if and only if the corresponding column in $D(x^*)$ contains only entries of the same sign or 0s. Since the type is partially identified if and only if the analyst can exclude an elementary type, the following is true:

Corollary 4. *The type is partially identified at x^* if and only if there is a column in $D(x^*)$ that contains both a positive and a negative entry.*

Corollary 4 provides an operational test for partial identification because we only need to compute matrix $D(x^*)$. For the next statement, we assume, as before, that x^* is a vertex of X at the intersection of N hyperplanes with linearly independent normal vectors. Clearly, the full type $I = F$ cannot be excluded in this case (and, therefore, any other type is fully identified only at $x^* \in \partial X$ that does not have this property):

Corollary 5. *The full type is always possible at any $x^* \in \partial X$ such that $\bar{B}(x^*)$ is an invertible $N \times N$ matrix. Hence, only the full type can be fully identified at such x^*.*

We can easily extend the results above to multiple observations. For example, Corollary 3 is extended as follows. Suppose that for all $t \in T$ and $x^t \in \partial X_t$, matrix $\bar{B}_t(x^t)$ is $N \times N$ and invertible. Then type I is possible at $\mathcal{O}$ if and only if there exists $a^I \in \mathbb{R}^N$ such that[16]

$$\mathbf{D}\mathbf{A}^{\mathbf{I}} \geq 0$$

for $\mathbf{D} = \left[D_1(x^1)\ldots D_{|T|}(x^{|T|})\right]$, where $D_t(x^*) = \left(\bar{B}_t^{\ T}(x^t)\right)^{-1}$, and $\mathbf{A}^{\mathbf{I}}$ consisting of $|T|$ diagonal blocks, each of which is a^I ($\mathbf{D}$ is $N \times N \cdot |T|$ and $\mathbf{A}^{\mathbf{I}}$ is $N \cdot |T| \times |T|$). Other results are extended similarly, including those that follow in the next subsection.

6.2 Finite sets

Our analysis has focused on convex sets. In some cases, however, the analysis for non-convex sets does not require fundamental changes. Consider the important case in which the feasible set consists of a finite number of alternatives represented by vectors $x^1, \ldots, x^k \in \mathbb{R}^N$. Conditions similar to those obtained previously can be derived in this environment, too. Let $\hat{X} = [x^1 \ldots x^k]$ be the corresponding $N \times k$ matrix.

Proposition 4. *Type I is possible at $x^* \in X$ such that $x^* \notin int\, conv(X)$ if and only if there exists $a^I \in \mathbb{R}^N$ such that*

$$E(x^*)\, a^I \geq \mathbf{0} \tag{4}$$

for the $k \times N$ matrix $E(x^*) = \left((x^*)\mathbf{1}^T - \hat{X}\right)^T$.

Note the similarity between (4) and (3) for polytopes. The intuition here is also similar: number $\left(E(x^*)\, a^I\right)_j$ measures the change in $\langle a^I, . \rangle$ when the choice changes from x^j to x^*. Analogous to Corollary 4, we have now the following criterion for partial identification:

[16]Notation $M \geq 0$ means here that each element of matrix M is non-negative.

Corollary 6. *With X finite, the type is partially identified at $x^* \in X$ such that $x^* \notin int\, conv(X)$ if and only if there is a column in $E\,(x^*)$ that contains both a positive and a negative entry.*

If i is the column with the opposite sign entries, then type $\{i\}$ can improve by moving away from x^* to some x^j irrespective of whether his evaluation of the relevant feature is positive or negative. Just like for a general convex set, the type is partially identified at x^* if and only if there is a type $\{i\}$ that is not possible at x^*. Note that the sign condition is also related to the orthant condition of Theorem 1. Indeed, the fact that the set of feasible directions straddles two different orthants can be interpreted as the existence of at least two feasible rates of exchange at x^* attaching opposite signs to the values of some feature. But the condition of Corollary 6 says exactly this, for movements from the chosen alternatives to two different ones.

Finally, observe that the caveat in the statement, $x^* \notin int\, conv(X)$, is necessary in a finite environment. This is because some interior points may be chosen by a non-linear type, while linear types always choose alternatives on the boundary. Hence, for $x^* \in int\, conv(X)$, there is no equivalence between the possible and linear types at x^*. Here, as well as in other non-convex cases, more radical changes might be required. This may be an interesting topic for future research.

7 Related literature

The classical view of alternatives as bundles of objectively defined characteristics, traceable to Lancaster (1966), assumes that all consumers perceive and value the same features. This assumption is problematic: different consumers may attend to different features, and analysts often cannot observe which features a given decision maker deems relevant.[17] Much of the literature in this vein has relied on specific functional forms to identify preferences. More recently, Blow et al. (2008) have pioneered a non-parametric, "revealed preference" type of analysis in such characteristics-based models. They characterise exactly which types of market choices by heterogeneous consumers are consistent with the model. Abaluck et al. (2023) study the problem of recovering preferences when consumers may be unaware of, or ignore, some of the characteristics of the goods they are consuming, but where they could search (at a cost) for more information on goods and

[17]Yet, the foundational issues raised by Lancaster—such as evaluating new goods and understanding complementarities—remain pressing. Ingenious solutions must be provided ad hoc (e.g., Gentzkow (2007)) when utility is defined on goods instead of features because, as Lancaster puts it, "there is no reason except "tastes"" why even wood and bread should not be close substitutes. Instead, the fact that objects such as of bread and wood are described very dissimilarly makes their non-substitutability intuitive within the theory. The problem, addressed by our analysis, is to turn such descriptions into observables.

discover its features; that is, in their framework agents may be inattentive to product characteristics, rather than products themselves. They provide sufficient conditions for preference estimation in a discrete choice setting, relying only on product feature data. Our approach shares their conceptual motivation, but departs methodologically: we adopt an abstract, choice-theoretic standpoint that neither presumes utility maximisation nor budget-constrained feasibility.

Viewed through this lens, our model bears affinity with Allen and Rehbeck (2023) who introduce *attribute variation*—rather than choice set variation—into stochastic choice models. They show that the class of strict perturbed utility models (PUMs) is fully characterised by a multivariate law of demand applied to utility indices and choice probabilities. Their work illustrates how variation in features can identify systematic preferences over attributes.

One natural explanation for selective attention to features is bounded rationality, particularly cognitive limitations. In this vein, Gabaix (2014) studies cognitively constrained agents who vastly simplify reality by efficiently allocating limited attention to different features. This leads them to perform "sparse optimisation", namely optimisation where only a few of the potentially relevant features are "switched on". Sparse econometric models embody similar insights, aiming to predict behaviour based on a small number of active features. In such cases our view of a DM is close to that of Gabaix, in that we admit DMs who may switch off many features. However, we remain agnostic as to whether this reflects a cognitive constraint or a principled stance—such as a political voter prioritising a single issue. Importantly, our approach does not presume sparsity; DMs may consider all features. Our framework fits the approach of Gabaix (2014) and the literature it has spawned; that of Demuynck and Seel (2018), who study consumers who focus their attention on a subset of the goods; and that of Chetty et al. (2009), who study consumers who may ignore the feature "sales tax" when making a purchase. It diverges, however, from Cerigioni and Galperti (2023), who explore how the order of attribute presentation affects valuation.

Recent interest in the theoretical foundations of survey design is also relevant to our analysis. We have highlighted the potential application of our framework to survey construction for identification purposes. A salient example is Apesteguia and Ballester (2023), who introduce a notion of rationalisability for binary survey responses, modelling opinions as points on a real line and linking response endorsement to proximity in attribute space.[18]

[18] Their concept of rationalisability for binary responses is that the DM's opinion can be expressed as a point on the real line in such a way that the DM endorses the questions that are closely aligned with the

At the technical level, our model presents points of contact with Che et al. (2024), who resolve a longstanding question about the relation between linear maximisation and strong Pareto optimality in convex choice sets (also discussed in Arrow et al. (1953)). They characterise the strong Pareto optimal frontier in terms of *sequential exposure,* namely the iterated application of linear maximisation to expose the face to whose interior a Pareto optimal point belongs. Some of the complexity in our characterisations comes precisely from the existence of points that are not (directly) exposed. Our example in Figure 1 is similar to the "canonical" example in Che et al. (2024), the main difference being that in our approach the directions of improvements are unknown. While we don't make use of their sequential exposure characterisation, it might be interesting, in future work, to distinguish types of different "orders". For example, at the choice point y in Figure 1 we might say that type $\{1\}$ is first-order, as its choice can be exposed in one-step by linear maximisation, whereas type $\{1, 2\}$ is "second-order", since its choice can only be exposed in two steps of linear maximisation. Nonetheless, one of the main issues for identification in our paper, the orientation of the set, is not relevant at all in Che et al. (2024), underscoring a core distinction in analytical focus.

8 Concluding remarks

Identifying the relevant features driving demand from limited data is notoriously difficult using standard econometric techniques. The conventional approach relies on observed demand shares and typically imposes structural assumptions on the utility function—such as additive separability—along with specific distributional assumptions on the unobserved heterogeneity (see, e.g., Berry and Pakes (2007)). By contrast, our abstract framework enables the identification of relevant characteristics even in highly data-sparse environments. Notably, we do not require the decision maker to maximise a utility function, nor do we rely on observed prices, as is the case in hedonic models. In this sense, our method offers a complementary perspective to standard empirical approaches.[19] Our approach can serve as an effective pre-screening device: when partial identification is possible, it may be used to eliminate irrelevant features before applying a more structured parametric model. This preliminary step could help ensure that subsequent estimation efforts focus only on characteristics with empirical support.

Importantly, our framework is designed to be applicable even in the extreme case

opinion.

[19]One restriction that our approach shares with this literature is the "monotonicity" assumption on characteristics and the lack of a "bliss point" (as they are either positive or negative). This may not be appropriate in some contexts, and an extension of the model in this direction would certainly be desirable.

where only a single observation is available. We have then showed how identification is enhanced as more observations accumulate.

Looking ahead, we are interested in two extensions of this approach. The first is to allow for non-monotonicities in the decision marker's evaluation of features. In the paper, we have assumed that each feature is either a positive or a negative, a modelling choice that parallels many parametric utility functions commonly used in applications. These functions typically lack interior bliss points, allowing a global distinction between goods and bads across the entire domain of choice. Furthermore, the assumption could be justified by limiting the analysis to a subdomain in which it is reasonable to assume monotonicity. Nonetheless, in settings where preferences are plausibly non-monotonic—such as the case of single-peaked preferences over public goods—extending the framework to accommodate such behaviour would be worthwhile.

The second extension is more radical. Suppose that the analyst observes a *distribution* p representing the shares of a finite number of choices. Clearly, when the DM's type is fully identified at all choices with a positive share within a given feasible set, the analyst can recover the distribution of types. Even if the type is only partially identified for some of the choices, the analyst can still recover the distribution of types up to a certain limit, by recovering bounds, implied by probability rules, on the possible type distributions. These bounds may, however, be relatively loose and thus offer limited precision. In such cases, additional stochastic observations from varied feasible sets can help refine the set of admissible type distributions—potentially leveraging techniques such as those proposed by Dardanoni et al. (2023).

Appendix A. Proofs

We first introduce a result that we will use repeatedly in the proofs below:

Lemma 1. *(i) If type I is linear at x^*, then I is possible at x^*. (ii) If type I is possible at $x^* \in X$, then there exists a type $J \subseteq I$ which is linear at x^*.*

Proof. (i) For any $v \in \mathbb{R}^N$, let $supp(v) = \{i \in F \mid v_i \neq 0\}$. Suppose that there exists $a \in \mathbb{R}^N$ such that $supp\,(a) = I$ and $\langle a, x^* \rangle \geq \langle a, x \rangle$ for all $x \in X$. Define an evaluation function e by setting $e\,(i) = 0 \iff a_i = 0$, $e\,(i) = 1 \iff a_i > 0$, and $e\,(i) = -1 \iff a_i < 0$. Suppose that for some $x \in X$, for all i, $e\,(i)\,x_i \geq e\,(i)\,x_i^*$. Then it must be $e\,(i)\,x_i = e\,(i)\,x_i^*$ for all i, in view of x^* being a maximiser of $\langle a, . \rangle$ on X. This means that x^* is e-admissible, and therefore I is possible at x^*.

(ii) Let I be possible at $x^* \in X$. First, consider the case where $I = F$. Since $x^* \in \partial X$, by the supporting hyperplane theorem, there exists a non-zero $a \in \mathbb{R}^N$ such that $\langle a, x^* \rangle \geq$

$\langle a, y\rangle$, for all $y \in X$. Then, the type $J = supp(a)$ is linear at x^* and satisfies $J \subseteq F$, where $J \subset F$ if $a_i = 0$ for some $i \in I$.

Next, suppose that $I \subset F$. In this case, consider the projection of all vectors $y \in X$ on the subspace V generated by $\{\mathbf{1}_i\}_{i\in I}$.[20] Let X_V denote this projection of all $y \in X$ on V. Since X is a compact, convex set, so is X_V. Observe that the projection x^*_V of x^* on V lies in ∂X_V, where ∂ denotes the boundary of X_V in the subspace V. To see this, suppose that $x^*_V \notin \partial X_V$. Then, for any evaluation function e with $e(i) \neq 0$, for all $i \in I$, there exists $y \in X_V$ with $e(i)\, y_i > e(i)\,(x^*_V)_i$, for all $i \in I$. But, this implies that I is not possible at $x^* \in X$ such that we have arrived at our desired contradiction. It follows that $x^*_V \in \partial X_V$. Therefore, by the supporting hyperplane theorem applied in V, there exists a non-zero $a \in \mathbb{R}^N$ such that $a_i = 0$, for all $i \in F \setminus I$, and $\langle a, x^*_V\rangle \geq \langle a, y\rangle$, for all $y \in X_V$. Clearly, it also holds that $\langle a, x^*\rangle \geq \langle a, y\rangle$, for all $y \in X$. Hence, the type $J = supp(a)$ is linear at x^* and satisfies $\varnothing \neq J \subseteq I$. □

Proposition 1

(i) Recall first that under the assumptions of the statement, for any non-zero $a \in \mathbb{R}^N$, there exists $x(a) \in \partial X$ such that $\langle a, x(a)\rangle \geq \langle a, y\rangle$ for all $y \in X$.[21] Thus, choosing a such that $supp(a) = I$, I is linear at $x(a)$. The conclusion follows from Lemma 1. (ii) By convexity, there exists a non-zero $a \in \mathbb{R}^N$ such that $\langle a, x\rangle \geq \langle a, y\rangle$ for all $y \in X$, and again Lemma 1 yields the conclusion with $I = supp(a)$.

Theorem 1

"only if": Suppose that $F_X(x^*) \subseteq K$ and that K is an orthant generated by $v^1, ..., v^N \in \mathbb{R}^N$. Then the orthant generated by $-v^1, ..., -v^N$ is contained in $N_X(x^*)$. In particular, for any $I \subseteq F$, $a^I = \sum_{i\in I}(-v^i) \in N_X(x^*)$. Then $\langle a^I, x^*\rangle \geq \langle a^I, x\rangle$ for all $x \in X$. By the first part of Lemma 1, type I is possible at x^*. Thus any type I is possible at x^* and the type is not identified.

"if": Suppose there is no orthant that contains $F_X(x^*)$. Then, there exist $v, w \in F_X(x^*)$ such that $v_i > 0$ and $w_i < 0$ for some $i \in \{1, \ldots, N\}$. Hence, $y_i > x^*_i > z_i$ for some $y, z \in X$. This implies that for any evaluation function e with $supp(e) = \{i\}$, x^* is not

[20] The projection of $y \in \mathbb{R}^N$ on the subspace V generated by $\{\mathbf{1}_i\}_{i\in I}$ is $z \in \mathbb{R}^N$ such that $z_i = y_i$ for $i \in I$ and $z_i = 0$ otherwise.

[21] We give the proof of this well-known fact here for completeness. Clearly, the function $f : X \to \mathbb{R}^N$, defined by $f(x) = \langle a, x\rangle$, $a \in \mathbb{R}^N$, is continuous. By Weierstrass Theorem, this implies that f has a maximiser $x(a) \in X$. If $x(a)$ is in the interior of X, then it is easy to show that $a = 0$. Since this case is excluded, $x(a) \in \partial X$.

e-admissible at x^* ($y_i > x_i^*$ excludes $e(i) = 1$, and $x_i^* > z_i$ excludes $e(i) = -1$). Therefore, type $\{i\}$ is not possible at x^* and the type is partially identified at x^*.

Theorem 2

"only if": Suppose there is an orthant K such that $\bigcup_t F\left(x^t\right) \subseteq K$. We can show using the construction in the first part of the proof of Theorem 1 that for any type I there exists e such that (e, I) is possible at all t. Hence, the type is not identified.

"if": Suppose there is no orthant K such that $\bigcup_t F\left(x^t\right) \subseteq K$. Then, there exist $v, w \in \bigcup_t F\left(x^t\right)$ such that $v_i > 0$ and $w_i < 0$ for some $i \in \{1, \ldots, N\}$. Clearly, $v \in F\left(x^t\right)$ and $w \in F\left(x^s\right)$ for some $t, s \in T$ (possibly with $t = s$). Hence, $(e, \{i\})$ is possible at both $\left(X_t, x^t\right)$ and $\left(X_s, x^s\right)$ only if $e(i) = -1$ (follows from $v_i > 0$) and $e(i) = 1$ (follows from $w_i < 0$), a contradiction. Therefore, type $\{i\}$ is not possible at $\mathcal{O}$ and the type is partially identified.

Theorem 3

Sufficiency. For Condition (i), suppose that (1) holds and let $I = \{i_1, \ldots, i_k\}$. We will show that any type $I' \neq I$ is not possible.

First, let I' be such that there exists $i' \in I'$ such that $i' \notin I$, and let e be an evaluation function for type I'. It follows from (1) that both $x^* + \varepsilon \mathbf{1}_{i'}$ and $x^* - \varepsilon \mathbf{1}_{i'}$ are in X for some $\varepsilon > 0$. If I' is possible, then either $e(i') = 1$ or $e(i') = -1$. Clearly, x^* is not e-admissible in the first case because $x^* + \varepsilon \mathbf{1}_{i'} \in X$, and it is not e-admissible in the second case because $x^* - \varepsilon \mathbf{1}_{i'} \in X$. Hence, I' is not possible.

Now assume there exists $I' \subset I$ that is possible with some evaluation function e. Assume also that $I = F$. The proof for $I \neq F$ is essentially identical but requires additional notation. Since $int\, K \cup \{0\}$ contains an orthant of $\mathbb{R}^N$, the next lemma shows that the projection of K on the subspace V generated by $\{\mathbf{1}_{i'}\}_{i' \in I'}$ coincides with the whole subspace V.

Lemma 2. *Let C_V be an orthant of $\mathbb{R}^N$ and K be a closed convex cone such that $C_V \subset int\, K \cup \{\mathbf{0}\}$. Then, for any $I' \subset F$, the projection of K on the subspace V generated by $\{\mathbf{1}_{i'}\}_{i' \in I'}$ coincides with V.*

Proof. Because K is a convex cone, we only need to show that $\mathbf{1}_{i'}$ and $-\mathbf{1}_{i'}$ belong to the projection for all $i' \in I'$. One of the two vectors $\mathbf{1}_{i'}$ and $-\mathbf{1}_{i'}$ lies in C_V and, hence, also in K by definition, so that it belongs to the projection. Without loss of generality, let this vector be $-\mathbf{1}_{i'}$. As for the other vector $\mathbf{1}_{i'}$, take any $j \notin I'$. By the condition of the lemma, there is a neighbourhood of v^j that is contained in K. This implies that there exists an $\varepsilon > 0$ such that $v^j + \varepsilon \mathbf{1}_{i'} \in K$. But then, the vector $d = \frac{1}{\varepsilon}(v^j + \varepsilon \mathbf{1}_{i'})$ is also in K. The projection of d on V is $\mathbf{1}_{i'}$. □

It follows from Lemma 2 that there exists $d \in K$ such that its projection $\bar{d}$ on V coincides with the evaluation function e, i.e., $\bar{d}_i = e(i)$ for all $i \in F$. Since by condition (1) $x^* + \varepsilon d \in X$ for some $\varepsilon > 0$, this implies that x^* is not e-admissible for I'. Therefore, I is the only possible type.

Sufficiency of (ii) is proven similarly. Note that $\dim X = N - 1$ implies that there is a unique (up to a non-zero scalar multiplication) vector $w \in \mathbb{R}^N$ that is orthogonal to all vectors in $F_X(x^*)$. Since w exposes X, we have $x^* \in \arg\max_{x \in X} \langle w, x \rangle$ and type $I = supp(w)$ is possible by Lemma 1. Let $I' \subseteq F$ be such that there exists $i' \in I'$ such that $i' \notin I$. Since $x^* \in ri\, X$, it must be that $x^* + e(i')\varepsilon \mathbf{1}_{i'} \in X$ for some $\varepsilon > 0$. But then x^* is not e-admissible for I'. Assume now that $I' \subset I$. Since $x^* \in ri\, X$, the projection of $F_X(x^*)$ on the subspace V generated by $\{\mathbf{1}_{i'}\}_{i' \in I'}$ coincides with V. Hence, there exists $d \in F_X(x^*)$ such that its projection on V is e. Since $x^* + \varepsilon d \in X$ for some $\varepsilon > 0$, this implies that x^* is not e-admissible for I'. Hence, such I' is not possible either.

Necessity. Suppose that the type is fully identified at $x^* \in \partial X$. In particular, this means that if $x^* \in \arg\max_{x \in X} \langle v, x \rangle$ and $x^* \in \arg\max_{x \in X} \langle w, x \rangle$ for $v, w \in \mathbb{R}^N$, then $supp(v) = supp(w)$. In the following two lemmas, we will use convex analysis to explore the restrictions this imposes on the structure of the normal and tangent cones at x^*.

Lemma 3. *The type is fully identified at $x^* \in \partial X$ only if either*

(i) $N_X(x^*) \subseteq ri\, C_{V_k} \cup \{\mathbf{0}\}$ *for some k-dimensional orthant* C_{V_k} *of* $\mathbb{R}^N$, $1 \leq k \leq N$,

or

(ii) $N_X(x^*)$ *is a 1-dimensional subspace of* $\mathbb{R}^N$.

Proof. Note that for any nonzero vector $x \in \mathbb{R}^N$, there exist a unique $1 \leq k \leq N$ and a unique k-dimensional orthant C_{V_k} such that $x \in ri\, C_{V_k}$. It follows from the definition of full identification and Lemma 1 that, since the type is fully identified, all non-zero vectors $a \in N_X(x^*)$ must share the same set of indices i with non-zero entries a_i. Thus, all non-zero vectors of $N_X(x^*)$ must belong to the relative interiors of the orthants of the same dimensionality k. If they all belong to the same k-dimensional orthant, then (i) holds. Otherwise, assume that there exist $x, y \in N_X(x^*) \setminus \{\mathbf{0}\}$ in different k-dimensional orthants. Since x and y share the same set of indices i with non-zero entries, this implies that $x_i > 0$

and $y_i < 0$ for some $i = 1, \dots, N$. But then, we have both $\alpha x_i + (1-\alpha) y_i = 0$ for some $\alpha \in (0,1)$, and $\alpha x + (1-\alpha) y \in N_X(x^*)$ by convexity. This contradicts the assumption of full identification unless $\alpha x + (1-\alpha) y = 0$. Thus, if x and y belong to different k-dimensional orthants, then they belong to the same 1-dimensional subspace. Since for a non-zero vector, there is only one 1-dimensional subspace containing it, Condition (ii) holds. The lemma is proved. □

Lemma 4. *The type is fully identified at $x^* \in \partial X$ only if either*

(i) $\dim X = N$ *and*

$$T_X(x^*) = \left\{ x \in \mathbb{R}^N \mid \left(x_{i_1}, \dots, x_{i_k}\right) \in K_T \right\}, \tag{5}$$

where $\{i_1, \dots, i_k\} \subseteq F$ are distinct indices, $1 \le k \le N$, and K_T is a closed convex cone in $\mathbb{R}^k$ such that int $K_T \cup \{\mathbf{0}\}$ contains an orthant of $\mathbb{R}^k$,

or

(ii) $\dim X = N - 1$ *and $x^* \in ri\, X$.*

Proof. First, consider the case $\dim X = N$. Let $I = \{i_1, \dots, i_k\}$ be the identified type. Note that for any $y \in \mathbb{R}^N$, there exist unique y_I and $y_{F \setminus I}$ such that $y = y_I + y_{F \setminus I}$, where $(y_I)_i = 0$ for all $i \in F \setminus I$ and $(y_{F \setminus I})_i = 0$ for all $i \in I$. Since I is the fully identified type, we have $\left\langle y_{F \setminus I}, x \right\rangle = 0$ for all $x \in N_X(x^*)$. Hence, $y \in T_X(x^*)$ if and only if $\langle y_I, x \rangle \le 0$ for all $x \in N_X(x^*)$. Clearly, such y_I's form a closed convex cone which we denote $T_{X,I}$.

We will show next that only Condition (i) of Lemma 3 is possible in the case $\dim X = N$. To the contrary, assume that $N_X(x^*)$ is a 1-dimensional subspace of $\mathbb{R}^N$. But then, because $N_X(x^*)$ and $T_X(x^*)$ are polar and $x, -x \in N_X(x^*)$ for some $x \neq 0$, we have $\langle y, x \rangle \le 0$ and $\langle y, -x \rangle \le 0$ for any $y \in T_X(x^*)$, which implies $\langle y, x \rangle = 0$. Thus, $T_X(x^*)$ is contained in the orthogonal complement[22] of $N_X(x^*)$ and, therefore, $\dim T_X(x^*) \le N - 1$. This contradicts the assumption $\dim X = N$. Thus, by Lemma 3, all the non-zero vectors of $N_X(x^*)$ are contained in the relative interior of a single k-dimensional orthant C_{V_k} of $\mathbb{R}^N$.

We will use the result of the previous paragraph to show that $ri\, T_{X,I} \cup \{0\}$ contains a k-dimensional orthant. Indeed, for any non-zero $x \in N_X(x^*)$, we can write $x = \sum_{n=1}^{k} \alpha_{i_n} v^{i_n}$, where $\alpha_{i_n} > 0$ for all $n = 1, \dots, k$, so that $\left\langle -v^{i_n}, x \right\rangle = -\alpha_{i_n} < 0$. Hence, $\langle y, x \rangle < 0$ for any

[22] The orthogonal complement $V^{\perp}$ of a subspace V of $\mathbb{R}^N$ is the subspace of all vectors $w \in \mathbb{R}^N$ that are orthogonal to all vectors $v \in V$, that is $V^{\perp} = \left\{ w \in \mathbb{R}^N \mid \langle w, v \rangle = 0 \ \forall v \in V \right\}$.

non-zero $y \in C_{V_k^-}$ and $x \in N_X(x^*)$, which implies that all the non-zero vectors of $C_{V_k^-}$ are contained in $ri\, T_{X,I}$. Then, the representation (5) follows.

Consider now the case $\dim X \leq N-1$. Note that there exists $y \in \mathbb{R}^N$ such that $y \neq 0$ and $\langle y, x - x^* \rangle = 0$ for all $x \in X$. This implies that both $y, -y \in N_X(x^*)$, so that only Condition (ii) of Lemma 3 remains possible. Thus, $\dim X \leq N-2$ is incompatible with full identification, since there exist at least two linearly independent vectors in $N_X(x^*)$ in this case. As for $\dim X = N-1$, observe that when x^* is on the relative boundary of X, there exists a supporting hyperplane containing x^* with the normal z in the same subspace. Since $z \in N_X(x^*)$ and also $y \in N_X(x^*)$ for some $y \neq 0$ such that $\langle y, z \rangle = 0$, Condition (ii) of Lemma 3 is violated. Therefore, $x^* \in ri\, X$. The lemma is proved. □

To conclude the proof of the theorem, it is left to show that (5) implies (1). Because the tangent cone is the closure of the cone of feasible directions, this does not follow automatically.

Let C_{V_k} be the orthant of $\mathbb{R}^k$ that is contained in $int\, K_T \cup \{\mathbf{0}\}$, i.e., for any $n \in \{1, \ldots, k\}$, we have $v^n \in \mathbb{R}^k$ and either $v^n = \mathbf{1}_n$ or $v^n = -\mathbf{1}_n$. For all $n \in \{1, \ldots, k\}$, let

$$\hat{v}^n = v^n - \varepsilon \sum_{\substack{n' \in \{1,\ldots,k\} \\ n' \neq n}} v^{n'} \tag{6}$$

(an example is shown in Figure 7). Since C_{V_k} is in $int\, K_T \cup \{\mathbf{0}\}$, there exists $\varepsilon > 0$ such that $C_{\hat{V}_k}$ (where $\hat{V}_k = \left\{\hat{v}^1, \ldots, \hat{v}^k\right\}$) is in $int\, K_T \cup \{\mathbf{0}\}$ too. Fix such ε and define K by

$$K = C_{\hat{V}_k}.$$

Clearly, K is a closed convex cone in $\mathbb{R}^k$ such that $int\, K \cup \{\mathbf{0}\}$ contains an orthant of $\mathbb{R}^k$. Define G by

$$G = \left\{ x \in \mathbb{R}^N \mid \left(x_{i_1}, \ldots, x_{i_k}\right) \in K \right\}.$$

By construction, we have $G \subseteq T_X(x^*)$. For the final step of the proof, we will show that $G \subseteq F_X(x^*)$.

Denote by I the identified type $\{i_1, \ldots, i_k\}$. If $I = F$, then $G \subseteq F_X(x^*)$ by the construction of G because in this case K is a cone in $\mathbb{R}^N$ and $K \subseteq F_X(x^*)$. Otherwise, let V be the subspace generated by $\{\mathbf{1}_{i'}\}_{i' \in F \setminus I}$. Define the slice S of $F_X(x^*)$ as

$$S = F_X(x^*) \cap V.$$

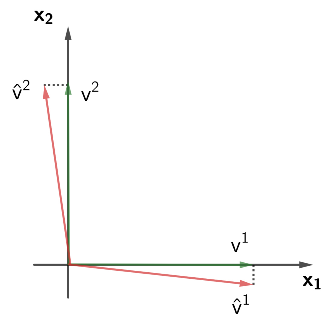

Figure 7: Example for equation (6)

Note that S contains all directions $y \in F_X(x^*)$ such that $y_i = 0$ for all $i \in I$. Clearly, S is a convex cone.

We will show that $S = V$. Assume, to the contrary, that this does not hold. Any convex cone that is not the whole space must be contained in a closed half-space. Applying this to V, there exists $v \in V$, $v \neq \mathbf{0}$, such that $\langle v, y \rangle \leq 0$ for all $y \in S$. Since $v \neq \mathbf{0}$, the set I' of indices $i' \in F$ for which $v_{i'} \neq 0$ is not empty. Then, we can show (Figure 8 contains an example) that type $I \cup I'$ is possible with

$$e(i) = \begin{cases} e_I(i) & \text{for } i \in I \\ 1 & \text{for } i \in I' \text{ such that } v_i > 0 \\ -1 & \text{for } i \in I' \text{ such that } v_i < 0 \\ 0 & \text{otherwise} \end{cases}$$

Indeed, if $x \in \mathbb{R}^N$, $e(i)x_i \geq e(i)x_i^*$ for all $i \in I \cup I'$, and $e(j)x_j > e(j)x_j^*$ for some $j \in I \cup I'$, then, for type I to be possible, it must be $x \notin X$ or $e(i)x_i = e(i)x_i^*$ for all $i \in I$. In the second case, we have $x - x^* \in V$ and $j \in I'$. Hence,

$$\langle v, x - x^* \rangle = \sum_{v_i > 0} v_i(x_i - x_i^*) + \sum_{v_i < 0} v_i(x_i - x_i^*) > 0$$

by the construction of e. This implies $x - x^* \notin S$ and $x \notin X$, from which it follows that $I \cup I'$ is a possible type. Since the possibility of both I and $I \cup I'$ contradicts full identification, we have $S = V$.

Therefore, $F_X(x^*)$ contains V, from which $G \subseteq F_X(x^*)$ follows by convexity. This concludes the proof of Theorem 3.

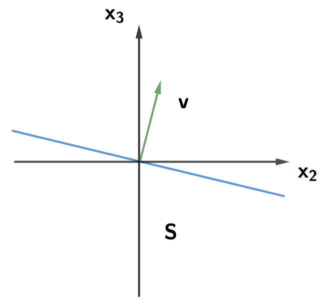

Figure 8: Example in the 23-plane. Possible types are 1 and 123

Theorem 4

Sufficiency. Obviously, all vectors $a \in ri\, S$ share the same set of indices i with non-zero entries a_i. Let $I \subseteq F$ be the set of all such indices. By Lemma 1, no type $J \subset I$ is possible at x^*. To see this observe that otherwise there would exist a type $K \subseteq J$ which is linear at x^*, contradicting that all vectors $a \in ri\, S$ share the same set of indices i with non-zero entries a_i. Now, consider all remaining types $J \subseteq F$, $J \neq I$. For such types, there exists $j \in J$ such that $j \notin I$. By $x^* \in ri\, Y$, where Y is a face of X parallel to $|F \setminus I|$ coordinate axes with $\dim Y \geq |F \setminus I|$, there exist $y^+, y^- \in X$ such that $y^+(j) > x^*(j) > y^-(j)$ and $y^+(i) = x^*(i) = y^-(i)$, for all $i \neq j$. It follows that no such type J is possible at x^*. Hence, type I is fully identified at x^*. A similar argument can be made for the non-zero vectors of a 1-dimensional subspace.

Necessity. In view of Lemma 3 in the previous proof, we only show the necessity of the second part of (i). Let I denote the type $\{i_1, \ldots, i_k\}$ identified at x^*. If $I = F$, then the second part of (i) is trivially satisfied. To see this, observe that then $k = |I| = |F| = N$. In this case, $x^* \in \mathrm{ri}\, Y$, for Y being a face of X with $\dim Y \geq N - N = 0$ always holds, because the relative interior of any point is the point itself. Analogously, the condition on Y being parallel to $N - N = 0$ coordinate axes also holds in a trivial way.

Next, let V be the subspace generated by $\{\mathbf{1}_{i'}\}_{i' \in F \setminus I}$ and Y be the face of X with $x^* \in Y$. Define the slice S of Y as

$$S = Y \cap V$$

Note that S contains all directions $y \in Y$ such that $y_i = 0$, for all $i \in I$. Clearly, S is a convex set.

We will show that $\mathbf{1}_{i'} \in V$ implies that both $(x^* + \varepsilon \mathbf{1}_{i'}) \in S$ and $(x^* - \varepsilon \mathbf{1}_{i'}) \in S$, for some $\varepsilon > 0$. Now, assume to the contrary, that this does not hold. Note that any convex set S for which this does not hold must be contained in a closed half-space of V that contains

x^*. As such, there exists $v \in V$, $v \neq \mathbf{0}$, such that $\langle v, y \rangle \leq 0$, for all $y \in S$. Since $v \neq \mathbf{0}$, the set I' of indices $i' \in F$ for which $v_{i'} \neq 0$ is not empty. Then, we can show that type $I \cup I'$ is possible using essentially the same argument used in the proof of Theorem 3. It follows that there exists $\varepsilon > 0$ such that for all $\mathbf{1}_{i'} \in V$ it holds that $(x^* + \varepsilon \mathbf{1}_{i'}) \in S$ and $(x^* - \varepsilon \mathbf{1}_{i'}) \in S$.

Hence, $x^* \in ri\, Y$, for some face Y of X parallel to $|F \setminus I| = N - k$ coordinate axes with $\dim Y \geq |F \setminus I| = N - k$. This concludes the proof.

Theorem 5

For sufficiency, the same argument of the proof of Theorem 3 can be used here with obvious adaptations, replacing (X, x^*) each time with a suitable element $\left(X_t, x^t\right)$ of the collection $\mathcal{O}$.

For the necessity proof, note that the intersection of the normal cones, $\bigcap_t N_{X_t}(x^t)$, must satisfy the conditions of Lemma 3: otherwise, multiple linear types would be possible at $\mathcal{O}$. Using this fact, we can show that the proof of Lemma 4 remains valid with $T_X(x^*)$ replaced by $conv\left(\bigcup_t T_{X_t}(x^t)\right)$. Indeed, the only non-trivial step (needed for the third paragraph of the proof of Lemma 4) is to show that

$$\left(\bigcap_t N_{X_t}(x^t)\right)^0 = conv\left(\bigcup_t T_{X_t}(x^t)\right), \tag{7}$$

To prove (7), note that, by the polar cone theorem (Bertsekas (2009), p. 100), we have

$$\left(C^0\right)^0 = cl\left(conv\left(C\right)\right)$$

for any nonempty cone C. By applying this equation to $C = \bigcup_t T_{X_t}(x^t)$, we get

$$\left(\left(\bigcup_t T_{X_t}(x^t)\right)^0\right)^0 = cl\left(conv\left(\bigcup_t T_{X_t}(x^t)\right)\right). \tag{8}$$

Clearly, the convex hull of a finite union of closed convex cones is closed, so we can omit the closure operator. Using the fact that the normal cone is the polar cone of the tangent cone, it is straightforward to check that

$$\left(\bigcup_t T_{X_t}(x^t)\right)^0 = \bigcap_t N_{X_t}(x^t).$$

By combining this with (8), we get the desired result (7). Therefore, the first condition of Lemma 4 holds with $T_X(x^*)$ replaced by $conv\left(\bigcup_t T_{X_t}(x^t)\right)$. The proof of the second condition is analogous with $x^* \in ri\, X$ replaced by $\mathbf{0} \in ri\, \mathcal{F}$. For the rest of the proof, we can use the same argument as in the final part of the proof of Theorem 3, replacing (X, x^*) with a suitable element of the collection $\mathcal{O}$.

Theorem 6

(i) Let types I and J be possible at x^*. We will show that $(e, I \cup J)$ is also possible at x^* with $e(i) = e_I(i)$ for $i \in I$, $e(i) = e_J(i)$ for $i \in J \setminus I$, and $e(i) = 0$ otherwise. For any evaluation function $\bar{e}$, let $supp(\bar{e}) = \{i \in F \,|\, \bar{e}(i) \neq 0\}$. Note that by definition $supp(e) = I \cup J$. Assume that $x \in \mathbb{R}^N$, $e(i)x_i \geq e(i)x_i^*$ for all $i \in I \cup J$, and $e(\hat{i})x_{\hat{i}} > e(\hat{i})x_{\hat{i}}^*$ for some $\hat{i} \in I \cup J$. We must show that $x \notin X$. The fact that (e_I, I) is possible implies that $x \notin X$ or $e(i)x_i = e(i)x_i^*$ for all $i \in I$. In the second case, we have $x_i = x_i^*$ for all $i \in I$, which implies $e_J(i)x_i = e_J(i)x_i^*$ for all $i \in I \cap J$. Hence, we have $\hat{i} \in J \setminus I$ and $e_J(i)x_i \geq e_J(i)x_i^*$ for all $i \in J$. Since (e_J, J) is possible, this implies $x \notin X$. Therefore, $I \cup J$ is a possible type.

(ii)[23] Let $\mathcal{I} \subseteq 2^F \setminus \{\varnothing\}$ be a nonempty collection of types that is closed under union. For each type $I \in \mathcal{I}$, let $\mathbf{1}_I \in \mathbb{R}^N$ be the vector whose i^{th} component is 1 if $i \in I$ and 0 otherwise (using the simplified notation $\mathbf{1}_i$ when $I = \{i\}$), and let $x_I = \frac{1}{|I|}\mathbf{1}_I$ (observe that $\langle x_I, \mathbf{1}\rangle = 1$). Also, let $x^i = \frac{1}{2}\mathbf{1}_i$. Note that, by the closure under union of $\mathcal{I}$, $supp\,(x) \in \mathcal{I}$ for all x in the convex hull $F^{\mathcal{I}}$ of the vectors x_I. We are going to construct the convex polytope X of the statement in various steps. First, let D be the downward closed[24] and convex hull of the vectors x_I such that $I \in \mathcal{I}$ and of the vectors $\left\{x^i\right\}_{i \in F}$. Next, to obtain a bounded set, let $Y = \{x \in D \,|\, x \geq -\mathbf{1}\}$. Note that the vectors x_I maximise on Y the linear function $\langle ., \mathbf{1}\rangle$ and $F^{\mathcal{I}}$ forms a face of Y. Moreover $\mathbf{0} \in int\, Y$ and Y is a convex polytope. It follows that the polar set $Y^* = \left\{x \in \mathbb{R}^N \,|\, \langle x, y\rangle \leq 1\, \forall y \in Y\right\}$ is also a convex polytope with $\mathbf{0} \in int\, Y^*$. Also, since $\langle \mathbf{1}, y\rangle \leq 1$ for all $y \in Y$, $\mathbf{1} \in Y^*$.

Now we set $X = Y^*$, and show that the set of types that are possible at $x^* = \mathbf{1}$ is $\mathcal{I}$. Take $I \in \mathcal{I}$. We show that (e, I) is possible at $\mathbf{1}$ with $e\,(i) = 1$ for all $i \in I$. In fact, if there is an x such that $x_i \geq 1$ for all $i \in I$ with $x_i > 1$ for some $i \in I$, then $\langle x, \mathbf{1}_I\rangle > \langle \mathbf{1}, \mathbf{1}_I\rangle$, which implies $\left\langle x, \frac{1}{|I|}\mathbf{1}_I\right\rangle > 1$, so that $x \notin X$.

Conversely, suppose that a type J is possible at $\mathbf{1}$. As will be shown later (Proposition 2), for a convex polytope a type that is possible at some x is linear at x. Therefore, $\mathbf{1}$ maximises some linear function $\langle c, .\rangle$, with $supp\,(c) = J$ and $\langle c, \mathbf{1}\rangle = 1$. Note that $c \in Y$.

[23] We are grateful to a referee for suggesting the argument in this proof.

[24] The downward closure of a set S is the set $\cup_{x \in X}\left\{y \in \mathbb{R}^N \,|\, y \leq x\right\}$.

Since $\langle y, \mathbf{1} \rangle < 1$ for all $y \in Y$ that are not in $F^{\mathcal{I}}$, $c \in F^{\mathcal{I}}$. Therefore, $J \in \mathcal{I}$.

Proposition 2

One direction is a special case of Lemma 1. For the other direction, recall that by a standard result (Arrow et al. (1953), Theorem 1) for a convex polytope $S \subset \mathbb{R}^N$ the set of points $x \in S$ such that $y \geq x \,\&\, y \neq x \implies y \notin S$ coincides with the set of points in S that maximise a linear function $\langle a, . \rangle$ where a has strictly positive components. A straightforward extension of this result is that, for an evaluation function e such that $e(i) \neq 0$ for all i, the set of e-admissible points coincides with the set of points in S that maximise a linear function $\langle a, . \rangle$ where a satisfies $sign(a_i) = e_i$ (the result in Arrow et al. (1953) corresponds to the case $e(i) = 1$ for all i). If (e, I) is possible at $x^* \in X$, then the projection x' of x^*on $\mathbb{R}^I$ has the property that $e(i) y_i \geq e(i) x'_i \, \forall i \in I \,\&\, y \neq x' \implies y \notin X'$, where X' denotes the projection of X on $\mathbb{R}^I$. What is more, $e(i) \neq 0$ for all $i \in I$. Since X' is also a convex polytope, by the previous result (applied to the subspace), there exists $a' \in \mathbb{R}^I$ with non-zero components such that $\langle a', x' \rangle \geq \langle a', y' \rangle$ for all $y' \in X'$. Therefore $\langle a^I, x \rangle \geq \langle a^I, y \rangle$ for all $y \in X$ where $a^I_i = a'_i$ for $i \in I$. Thus, I is a linear type at x.

Proposition 3

Suppose that type I is possible at x^*. By Proposition 2, this means that there exists an $a^I \in \mathbb{R}^N$ such that x^* is an optimal solution to the linear program $\max_{Bx \leq c} \langle a^I, x \rangle$. By the strong duality theorem (Bertsimas and Tsitsiklis (1997), p. 148), the dual problem $\min_{B^T y = a^I, y \geq 0} \langle c, y \rangle$ also has an optimal solution y^*. Since y^* is feasible, we have $y^* \geq \mathbf{0}$ and $a^I = B^T y^*$. By complementary slackness (Bertsimas and Tsitsiklis (1997), p. 151),

$$y_j^* \left(B^{(j)} x^* - c_j \right) = 0, \text{ for all } j = 1, \ldots, m,$$

where $B^{(j)}$ is the j-row of B, $j = 1, \ldots, m$. Therefore, $y_j^* = 0$ for the constraints that are not active at x^*. This implies that $a^I = \bar{B}^T (x^*) \bar{y}^*$ for $\bar{y}^*$ obtained from y^* by filtering out non-active constraints.

Conversely, suppose that $a^I = \bar{B}^T (x^*) y$ for some $y \geq \mathbf{0}$ and $a^I \in \mathbb{R}^N$. Let $y_j^* = y_j$ if the constraint j is active at x^* and $y_j^* = 0$ otherwise. Since $y^* \geq \mathbf{0}$ and $a^I = B^T (x^*) y^*$, y^* is a feasible solution of the dual problem. By complementary slackness, x^* is an optimal solution of the primal problem $\max_{Bx \leq c} \langle a^I, x \rangle$, which implies that type I is linear, and therefore possible, at x^*.

Proposition 4

Obviously, a type that is possible at x^* is linear in this case as well. Then $\left(a^I, I\right)$ is possible if and only if for any $x \in X$, we have $\left\langle a^I, x^*\right\rangle \geq \left\langle a^I, x\right\rangle$, which is equivalent to $\left\langle a^I, x^* - x\right\rangle \geq 0$, from which condition (4) follows.

Appendix B. Convex analysis

For the reader's convenience we gather here some standard facts and terminology used in the text. The *affine hull* $aff\,(S)$ (resp., *convex hull*) of a set $S \in \mathbb{R}^N$ is the smallest affine (resp., convex) set containing S, namely

$$aff\,(S) = \left\{ \sum_{i=1}^{k} \alpha_i x^i \,|\, k > 0,\ x^i \in S,\ \alpha_i \in \mathbb{R},\ \sum_{i=1}^{k} \alpha_i = 1 \right\},$$
$$conv\,(S) = \left\{ \sum_{i=1}^{k} \alpha_i x^i \,|\, k > 0,\ x^i \in S,\ \alpha_i \geq 0,\ \sum_{i=1}^{k} \alpha_i = 1 \right\}.$$

The *dimension* $\dim S$ of a set S is the dimension of its affine hull. The *relative interior* $ri\,S$ of a set S is the interior which results when S is regarded as a subset of its affine hull. For a convex set S in $\mathbb{R}^N$, $x \in S$ belongs to $ri\,S$ iff for all $y \in S$ there exists $\lambda > 1$ such that $\lambda x + (1-\lambda)\, y \in S$. The *face* E of a convex set S is a nonempty convex subset of S such that: if $x \in E$ and $x = \alpha y + (1-\alpha)\, z$ for some $y, z \in S$ and $\alpha \in (0,1)$, then $y, z \in E$ (i.e. no element of the face can be obtained as a convex combination of elements outside the face). A zero–dimensional face is an *extreme* point. That is, $x \in S$ is extreme if there exists no $y, z \in S \setminus \{x\}$ and $\alpha \in (0,1)$ such that $x = \alpha y + (1-\alpha)\, z$. A face E of S is *exposed* if there is a $w \in \mathbb{R}^N$ such that $E = \arg\max_{x \in S} \langle w, x\rangle$; if so we say that w exposes E. In particular, a point $x \in S$ is exposed if $x = \arg\max_{x \in S} \langle w, x\rangle$ for some $w \in \mathbb{R}^N$.

A *cone* is a set C such that if $x \in C$ and $\alpha \geq 0$ then $\alpha x \in C$. The *cone of feasible directions* of a set S at $x \in S$ is defined as[25]

$$F_S\,(x) = \{\alpha\,(y - x) \,|\, y \in S,\ \alpha \geq 0\}.$$

The *tangent cone* at $x \in S$ is defined as $T_S\,(x) = cl\,F_S\,(x)$, where cl denotes the closure operator.[26] The tangent cone is a useful tool to describe the structure of a set S by means

[25] A more general definition of the cone of feasible directions is $F_S\,(x) = \left\{y \in \mathbb{R}^N | \,\exists \varepsilon > 0 \text{ such that } x + \varepsilon y \in S\right\}$, but the two definitions coincide given that S is convex.

[26] See, e.g., Rockafellar and Wets (1997), Theorem 6.9, for a proof that for convex sets more general definitions of the tangent cone reduce to this one.

of the feasible directions *and* their limits. The *normal cone* of S at $x \in S$ is defined as

$$N_S(x) = \left\{ z \in \mathbb{R}^N | \langle z, y - x \rangle \leq 0, \forall y \in S \right\}.$$

Intuitively, the normal cone at the point $x \in S$ is the set of all vectors that make an acute angle with no vector from x to some point y in the feasible set. When $S \subseteq \mathbb{R}^N$ is a compact, convex set and $a \in \mathbb{R}^N$, the following fact holds: $x^* \in \arg\max_{x \in S} \langle a, x \rangle$ *if and only if* $a \in N_S(x^*)$.

The *polar cone* S^0 of S is defined as

$$S^0 = \left\{ y \in \mathbb{R}^N | \langle y, x \rangle \leq 0, \forall x \in S \right\}$$

and its negative $-S^0$, which comprises all vectors that form non-obtuse angles with any vector in S, is called the *dual cone*. When S is convex, the normal and the tangent cones at x are polar to each other.

A *convex polytope* is a set $S = conv\left(x^1, ..., x^N\right)$ where the x^i are the *vertices*.[27] A *polyhedron* is a set $S = \{x \in \mathbb{R}^N \mid Bx \leq c\}$ where B is an $m \times N$-matrix and c is a vector in $\mathbb{R}^m$, with m finite. A polyhedron S is bounded if there is some $k > 0$ such that $||x|| \leq k$ for all $x \in S$, where $|| \cdot ||$ denotes the Euclidean norm. A set is a bounded polyhedron if and only if it is a convex polytope.

The *polar set* of a set S is $S^* = \left\{ y \in \mathbb{R}^N | \langle y, x \rangle \leq 1 \, \forall x \in S \right\}$. If S is a closed convex set such that $\mathbf{0} \in S$, then $\left(S^*\right)^* = S$. If S is a convex polytope and $\mathbf{0} \in int\, S$ then S^* is also a convex polytope and $\mathbf{0} \in int\, S^*$.

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[27] When $S = \left(x^1, ..., x^N\right)$ we denote $conv\,(S)$ as $conv\left(x^1, ..., x^N\right)$.

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